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Kolmogorov inertial range spectra for inhomogeneous turbulence

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It is shown that the statistical orthogonality of the Karhunen–Loeve (KL) eigenfunctions with respect to both energy and dissipation makes them a particularly good basis for the definition of an energy spectrum in inhomogeneous fluid flows. An effective wave number is defined to characterize the KL eigenfunctions. The definition preserves the relationship between the dissipation and energy spectra that holds for Fourier spectra. With the spectrum and wave number so defined, the scale-similarity arguments that lead to the existence of a spectral inertial range apply. It is also shown that the existence of a spectral inertial range in the KL eigenspectrum is consistent with Kolmogorov's scale-similarity formulation for structure functions. An example of the KL spectrum obtained from a numerically simulated plane channel flow is presented.

I. INTRODUCTION

The scale similarity theory of Kolmogorov[1] is one of the most successful in turbulence. The $k^{-5/3}$ inertial range spectrum it predicts has been observed in many experiments, and indeed the Kolmogorov scaling laws are so well established that new theories are required to be compatible with them. Though Kolmogorov originally applied his arguments to the statistics of velocity differences at nearby points in the flow, the theory has most often and most successfully been applied to the Fourier spectrum. However, analysis of the spatial Fourier spectrum is strictly applicable only in the case of homogeneous turbulence. Several authors (e.g., Monin and Yaglom[2] and Batchelor[3]) make the argument, as did Kolmogorov, that at high Reynolds numbers even inhomogeneous turbulence is "locally homogeneous." They then go on to suggest that in some sense a "local" Fourier analysis at high wave numbers can be applied for which the inertial range theory would hold. This is an appealing argument, but the Fourier representation is formally global, and the sense in which it can be considered local is not defined (see Sec. III). Thus, for inhomogeneous flows, the application of spectral inertial-range theory awaits the identification of an appropriate functional basis, analogous to the Fourier functions, by which the spectrum can be defined. In this paper we seek a *global* functional basis. One could also consider local basis functions (e.g., wavelets), but they are not considered here.

In a recent letter by Knight and Sirovich,[4] it was suggested that the Karhunen–Loeve (KL) eigenfunctions[5] (empirical eigenfunctions) were a particularly good functional basis for measuring inertial-range spectra in inhomogeneous flows. This suggestion was based on several examples in which KL eigenvalue spectra of low-Reynolds number turbulent flows exhibited apparent inertial ranges. Later, Sirovich[6] speculated that this may be due to the fact that the KL eigenfunctions provide an optimum representation of the energy in a turbulent flow.

In this paper Knight and Sirovich's suggestion is examined theoretically to determine if and why the KL eigenfunctions are an appropriate basis for defining an energy spectrum, and, in particular, for obtaining a spectral inertial range. To do this, several well-known properties of the KL eigenfunctions will be needed. They are stated below without proof (see Ref. 5 and the references therein for details).

A. Karhunen–Loeve eigenfunctions

For a flow in a bounded domain $\mathscr{D}$, the (possibly complex, vector) KL eigenfunctions $\phi_i(\mathbf{x})$ are solutions of the integral equation:

$$\int_{\mathscr{D}} R_{ij}(\mathbf{x},\mathbf{x}')\phi_j(\mathbf{x}')d\mathbf{x}'=\lambda\phi_i(\mathbf{x}), \tag{1}$$

where $R_{ij}(\mathbf{x},\mathbf{x}')=\langle u_i(\mathbf{x})u_j(\mathbf{x}')\rangle$ is the two-point correlation tensor of the fluctuating velocity u_i ($\langle\ \rangle$ signifies the expected value and the velocities are at the same time), λ is the eigenvalue, and summation over repeated tensor indices is assumed. There are countably infinite solutions $\phi_i^{(n)}$ and eigenvalues λ_n. Due to the symmetry properties of the kernel R_{ij}, the eigenvalues λ_n are real and non-negative, and are usually enumerated in order of decreasing magnitude. It is convenient (though not necessary) to consider the eigenfunctions to be complex, in general, so that the complex exponential can be used (see below). The eigenfunctions are orthogonal, and we will assume them to be normalized such that the integral of their magnitude squared is unity; thus

$$\int_{\mathscr{D}} \phi_i^{(n)}\phi_i^{(m)*}\,d\mathbf{x}=\delta_{mn}, \tag{2}$$

where $*$ denotes the complex conjugate. The eigenfunctions form a complete set for fields satisfying the velocity boundary conditions, and thus can be used as a basis to represent the velocity:

$$u_i(\mathbf{x})=\sum_{n=1}^{\infty} a_n\phi_i^{(n)}(\mathbf{x}). \tag{3}$$

In (3), the complex a_n are uncorrelated random variables (stochastic processes in time) with variance λ_n, that is

$$\langle a_n a_m^* \rangle = \lambda_m \delta_{mn}. \tag{4}$$

The eigenvalue λ_n is twice the average energy contributed by mode n of the decomposition, and

$$\int_{\mathscr{D}} \langle u_i u_i \rangle d\mathbf{x} = \sum_{n=1}^{\infty} \lambda_n. \tag{5}$$

In the special case in which the turbulence is homogeneous in one or more spatial directions (with periodic boundary conditions), the eigenfunctions have a complex exponential functional dependence in those directions. In this case the KL decomposition reduces to the Fourier decomposition. Finally, $\phi_i^{(1)}$ is the normalized function that maximizes the projection of u_i, that is, it maximizes

$$\langle |a_1|^2 \rangle = \left\langle \left| \int_{\mathscr{D}} u_i \phi_i^{(1)*} \, d\mathbf{x} \right|^2 \right\rangle, \tag{6}$$

and $\phi_i^{(1)}$ and $\phi_i^{(2)}$ are the two functions that maximize $\langle |a_1|^2 \rangle + \langle |a_2|^2 \rangle$ and so forth for all n. Thus the ϕ_i expansion produces the most rapid convergence of partial sums to the energy.

Other sets of eigenfunctions and eigenvalues can be obtained by replacing R_{ij} in (1) with $R_{ij}^{\mathscr{L}} = \langle \mathscr{L}[u_i(\mathbf{x})] \mathscr{L}[u_j(\mathbf{x}')] \rangle$, where $\mathscr{L}$ is any linear operator. In this case all the previous results hold with u_i replaced with $\mathscr{L}(u_i)$. A truncated expansion like (5) is then an optimum representation of the quantity $\int_{\mathscr{D}} \mathscr{L}(u_i) \mathscr{L}(u_i) d\mathbf{x}$. This amounts to selecting the norm and inner product with which the KL analysis is to be performed.

Finally, as suggested by the above discussion, the KL eigenfunctions in an inhomogeneous flow are not known *a priori*. Thus, unlike homogeneous flows, inhomogeneous flows require more than just the KL spectrum to describe the second-order moments. Furthermore, it is difficult to use the KL spectrum as an experimental or computational diagnostic for inhomogeneous flows because the complete two-point correlation is generally needed to compute it. These limitations arise because inhomogeneous flows are inherently more difficult to describe and understand than homogeneous flows.

In Sec. II, the classical arguments leading to the inertial range spectrum are generalized to inhomogeneous flows. A derivation of an inhomogeneous inertial range spectrum from Kolmogorov's original similarity hypotheses is developed in Sec. III. An analysis of plane channel flow is presented as an example in Sec. IV, and is followed by concluding remarks in Sec. V.

II. THE INERTIAL-RANGE SPECTRUM

In homogeneous isotropic turbulence, the energy spectrum is easily defined from the spectrum tensor, which is the Fourier transform of the two-point correlation tensor,

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{8\pi^3} \int R_{ij}(\boldsymbol{\delta}) e^{-i\mathbf{k}\cdot\boldsymbol{\delta}} \, d\boldsymbol{\delta}, \tag{7}$$

where $\mathbf{k}$ is the wave vector (k_1, k_2, k_3) and here the two-point correlation only depends on $\boldsymbol{\delta} = \mathbf{x}' - \mathbf{x}$. The energy density in the three-dimensional wave space is $\mathscr{E}(\mathbf{k}) = \frac{1}{2}\Phi_{ii}(\mathbf{k})$, which, due to isotropy, must depend only on the magnitude k of the wave vector $\mathbf{k}$. Thus, the three-dimensional energy spectrum $E(k)$ is obtained by integrating $\mathscr{E}(\mathbf{k})$ over all $\mathbf{k}$ with given magnitude k to obtain

$$E(k) = 4\pi k^2 \mathscr{E}(k). \tag{8}$$

Equivalently, we could start with the Fourier transform of the velocity itself,

$$\hat{u}_i(\mathbf{k}) = \frac{1}{8\pi^3} \int u_i e^{-i\mathbf{k}\cdot\mathbf{x}} \, d\mathbf{x}, \tag{9}$$

and define the spectrum tensor as $\Phi_{ij} = \langle \hat{u}_i \hat{u}_j^* \rangle$.

The inertial-range spectrum was first obtained theoretically by Obukov,[7] but his derivation relied on a model for the transfer of energy from small wave numbers to large wave numbers. The more fundamental derivation relies directly on scale-similarity arguments like those of Kolmogorov,[1] and can be found in any text on turbulence.[2,3] They are repeated here so that we may examine them to see how they can be extended to inhomogeneous flows.

Following Kolmogorov,[1] we identify the bulk rate of energy dissipation (rate of energy transfer across the spectrum),

$$\epsilon = \frac{\nu}{2} \left\langle \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\rangle = 2\nu \int_0^{\infty} k^2 E(k) dk, \tag{10}$$

along with the kinematic viscosity ν as the important dimensional parameters characterizing the turbulence. Note that (10) allows us to define the dissipation spectrum $D(k) = 2\nu k^2 E(k)$, describing the dissipation associated with each wave number. Also required are a velocity (U) and length (L) characterizing the largest scales in the flow (e.g., $\langle u_i u_i \rangle^{1/2}$ and the integral scale). From strictly dimensional arguments, we then have

$$E(k) = \epsilon^{2/3} k^{-5/3} F(k\eta, kL, UL/\nu), \tag{11}$$

where $\eta = (\nu^3/\epsilon)^{1/4}$ is the Kolmogorov length scale and F is a universal function. The length scales L and η are the scales of motion associated with the energy and dissipation in the flow. Kolmogorov's first similarity hypothesis[1] states that if the scale of motion is sufficiently small relative to the energetic scales ($kL \gg 1$) then $E(k)$ cannot depend on the large-scale parameters L and U, so in this regime, F is a function of $k\eta$ alone. The second similarity hypothesis states that if the Reynolds number UL/ν is sufficiently large, so that, in addition, the scale of motion is large enough relative to the dissipative scales ($k\eta \ll 1$) that the dissipation associated with it is negligible [$D(k)$ is small], then $E(k)$ cannot depend explicitly on the viscosity (or η). In this case F must be a (universal) constant C_1. Thus, we obtain the well-known inertial-range spectrum:

$$E(k) = C_1 \epsilon^{2/3} k^{-5/3}, \tag{12}$$

which can only exist if the Reynolds number is so high that the energy containing and dissipation scales are well separated. Note that (12) implies that $\mathscr{E}(k) = (C_1/4\pi)\epsilon^{2/3} k^{-11/3}$. For simplicity we are not considering here the intermittency of the dissipation pointed out by Landau,[8] but the arguments below apply as well to the refined theories developed to account for intermittency (see Ref. 2, for example).

Two properties of the Fourier expansions underlying the spectra discussed above were important in these arguments. First, Parseval's relation holds for both energy and dissipation (and any other quadratic quantity), so that each individual mode makes a distinct positive-definite contribution to the energy and the dissipation. This partitioning of the energy between the modes is important because otherwise $E(k)$ could not be interpreted as an energy spectrum and would therefore be of no interest. Partitioning of the dissipation between the modes is important because the second similarity hypothesis required that the *dissipation* associated with a mode (or wave number) be negligible. If Parseval's relation did not hold for dissipation, the negligible modal dissipation condition would not be obvious (see the discussion below). Second, the dimensional analysis relied on the fact that each Fourier mode is characterized by a well-defined length-scale ($1/k$). For Fourier functions this length scale characterization is unambiguous, since different Fourier functions are identical, except for their scale.

Now consider the expansion of the velocity in a finite domain ($\mathscr{D}$) in some arbitrary (complete) basis set, which, in general, is neither orthogonal or scale similar;

$$u_i(\mathbf{x})=\sum_{j=1}^{\infty} a_j\psi_i^{(j)}(\mathbf{x}). \tag{13}$$

The energy $\bar{e}$ and dissipation $\bar{\epsilon}$ in the domain are then

$$\bar{e}=\left\langle\frac{1}{2}\int_{\mathscr{D}} u_iu_i\,d\mathbf{x}\right\rangle$$

$$=\frac{1}{2}\sum_n\sum_m\langle a_na_m^*\rangle\int_{\mathscr{D}}\psi_i^{(n)}\psi_i^{(m)*}\,d\mathbf{x}, \tag{14a}$$

$$\bar{\epsilon}=\left\langle 2\nu\int_{\mathscr{D}}\mathscr{S}_{ij}(\mathbf{u})\mathscr{S}_{ij}(\mathbf{u})d\mathbf{x}\right\rangle$$

$$=2\nu\sum_n\sum_m\langle a_na_m^*\rangle\int_{\mathscr{D}}\mathscr{S}_{ij}(\psi^{(n)})\mathscr{S}_{ij}(\psi^{(m)})^*\,d\mathbf{x}, \tag{14b}$$

where $\mathscr{S}_{ij}(\mathbf{u})=\frac{1}{2}(\partial u_i/\partial x_j+\partial u_j/\partial x_i)$. Parseval's relation for energy and dissipation holds in the case of the Fourier functions because they satisfy

$$\int_{\mathscr{D}}\psi_i^{(n)}\psi_i^{(m)*}\,d\mathbf{x}=\delta_{nm} \tag{15a}$$

and

$$\int_{\mathscr{D}}\mathscr{S}_{ij}(\psi^{(n)})\mathscr{S}_{ij}(\psi^{(m)})^*\,d\mathbf{x}=\epsilon_n\delta_{nm}, \tag{15b}$$

where ψ has been normalized as in (2), there is no summation in n, and $\epsilon_n=\int_{\mathscr{D}}\mathscr{S}_{ij}(\psi^{(n)})\mathscr{S}_{ij}(\psi^{(n)})^*\,d\mathbf{x}$. However, only the Fourier functions (in particular domains) satisfy both conditions in (15), since they are the eigenfunctions of the derivative operator. The Fourier functions are not appropriate for inhomogeneous flows, as discussed in the Introduction, so we cannot expect both conditions in (15) to be satisfied. Fortunately, partitioning of the energy and dissipation into individual modes also occurs if

$$\langle a_na_m^*\rangle=\delta_{nm}\langle|a_n|^2\rangle \tag{16}$$

(no summation with respect to n). This is, in some sense, a weaker condition than (15), since only *statistical* orthogonality holds, rather than the functional orthogonality implied by (15).

As discussed above, the definition of an energy spectrum requires that energy be partitioned between the modes. Thus to pursue our generalization of the inertial range spectrum, we require that the ψ expansion functions satisfy either (15a) or (16). We must also generalize the requirement of negligible dissipation for the second similarity hypothesis. One way to do this is to consider the dynamic equation for the energy in a mode. In the case of Fourier expansions, the viscous term in this equation is just the dissipation associated with the mode. The negligible dissipation requirement is thus seen to be a requirement that the dynamics of modal energy be unaffected by viscosity. With general expansion functions as in (13), the viscous term in the equation for $E_n=\langle|a_n|^2\rangle/2$ (the energy in the mode n) is given by

$$D_n=-2\nu\sum_m\langle a_na_m^*\rangle\int_{\mathscr{D}}\mathscr{S}_{ij}(\psi^{(n)})\mathscr{S}_{ij}(\psi^{(m)})^*\,d\mathbf{x}, \tag{17}$$

which we thus require to be negligible for modes in an inertial range. Equation (17) is obtained by integration by parts, where the boundary terms are assumed to be zero. The boundary terms will be zero in the case of no-slip boundaries (for example). It is not clear how this argument can be extended to cases in which the boundary terms are not zero.

The inertial range occurs over a range of the length scales characterizing the expansion functions ($1/k$ in the case of Fourier). The definition of these length scales is not trivial here since the expansion functions are not generally scale similar, as the Fourier functions are. For the purposes of inertial-range theory, the length-scale characterization must be chosen such that the magnitude of the dissipation (D_n) relative to the energy (E_n) of a mode increases with decreasing length scale. This ensures that "large"-scale modes will be energetic but not dissipative, while "small"-scale modes will be dissipative but not energetic, as required by the inertial-range arguments. The most direct way to accomplish this is to make the length scale depend explicitly on the modal dissipation (D_n) and the modal energy (E_n). Furthermore, a sensible length scale characterization must depend only on the shape of the expansion function ($\psi^{(n)}$), not on its amplitude or on other quantities (e.g., the amplitudes of other modes). The only length scale (expressed here as the effective wave number k_n) satisfying these restrictions, is given by

$$k_n^2=\frac{D_n}{2\nu E_n}=2\epsilon_n, \tag{18}$$

where ϵ_n is as defined in (15). The coefficient of $\frac{1}{2}$ is arbitrary, and is chosen by analogy to the equivalent relationship for Fourier functions. Furthermore, for (18) to be valid, we require that the ψ expansion satisfy either (15b) or (16), so that k_n depends only on the shape of $\psi^{(n)}$, and $D_n = 2\nu\epsilon_n\langle |a_n|^2\rangle$. Thus, the expansion must partition the dissipation as well as the energy. One could certainly devise expansion functions and associated length scales that did not satisfy (18), but, which nonetheless segregated the energy and dissipation into the large and small scales, respectively. But, this would require that the functions and length scales allow the relative magnitude of the dissipation to be related to or at least bounded by the wave number squared.

As was noted above, only the Fourier functions satisfy both conditions (15a) and (15b). Thus if the expansion functions $\psi^{(n)}$ are to produce a partitioning of both the energy and the dissipation (similar to Parseval's relation), as discussed above, then the expansion coefficients must satisfy (16). This condition is only satisfied by the KL eigenfunctions (4) and the KL eigenfunctions based on $R^{\mathscr{L}}$ for linear $\mathscr{L}$. Thus, it appears that the KL eigenfunctions are indeed a particularly good choice of expansion functions for general inhomogeneous flows, as suggested by Knight and Sirovich.[4]

The above discussion suggests that the similarity arguments leading to an inertial-range spectrum for homogeneous turbulent flows are valid for the KL spectrum of any sufficiently high Reynolds number turbulent flow, given the generalized wave number, as defined in (18). Other more involved derivations of the inertial-range spectrum based on models of the transfer spectrum (e.g., Obukov[7] and Heisenberg[9]) also rely on statistical orthogonality with respect to energy and dissipation, so the KL spectrum would be good for these derivations as well.

III. CONSISTENCY WITH KOLMOGOROV'S ORIGINAL DEVELOPMENT

As mentioned in the Introduction, Kolmogorov did not consider the energy spectrum itself, but rather the joint probability distribution of the difference in velocity at nearby points in the flow. This has the advantage that it is directly applicable to inhomogeneous flows. When the velocity differences are homogeneous and isotropic for sufficiently small spatial separations, Kolmogorov's similarity hypotheses yield

$$R_{ij}(\mathbf{x},\mathbf{x}') \approx R_{ij}(\tilde{\mathbf{x}},\tilde{\mathbf{x}}) - \tfrac{2}{3}C\epsilon^{2/3}(\tilde{\mathbf{x}})r^{2/3}[\delta_{ij} - \tfrac{1}{4}\cos(\theta_i)\cos(\theta_j)], \qquad (19)$$

provided that $r/L \ll 1$ and $r/\eta \gg 1$, where $r = |\mathbf{x}' - \mathbf{x}|$, $\tilde{\mathbf{x}} = (\mathbf{x}' + \mathbf{x})/2$, and $\cos(\theta_i) = (x_i' - x_i)/r$. Using this result, we should be able to show that the KL spectrum in an inhomogeneous flow exhibits an inertial range, in the same way that (19) can be used to show that the Fourier spectrum has an inertial range for homogeneous isotropic turbulence. The existence of local isotropy is currently controversial,[10–12] but the isotropic form (19) is used below only for simplicity. A similar analysis could be done using a more general nonisotropic form.

A. One-dimensional spectrum

For the sake of simplicity, consider a flow that is homogeneous in two spatial directions (say x_1 and x_3), and the KL eigenfunctions representing the $x_2 = y$ variation of the single velocity component u_1 in a finite domain $y \in [a,b]$. The more complicated problem of the three-dimensional representation of the velocity is discussed in Sec. III B. In the simple one-dimensional case, (19) reduces to

$$R_{11}(y,y') \approx R_{11}(\tilde{y},\tilde{y}) - \tfrac{2}{3}C\epsilon^{2/3}(\tilde{y})|y - y'|^{2/3}, \qquad (20)$$

where $\tilde{y} = (y' + y)/2$, and the KL eigenfunctions satisfy

$$\int_a^b R_{11}(y,y')\phi(y')dy' = \lambda\phi(y). \qquad (21)$$

Following Sirovich and Knight,[13] we seek the asymptotic behavior of the eigenfunctions as $\alpha \to \infty$ in the WKB form:

$$\phi(y) = A(y)e^{i\alpha g(y)}, \qquad (22)$$

where $dg(y)/dy > 0$ for $y \in [a,b]$, $\int_a^b |A|^2\,dy = 1$, and $1/(b-a)\int_a^b (dg/dy)^2\,dy = 1$ (normalization conditions), and it is assumed that derivatives of g and A are of order 1 or smaller in α. Nonzero dg/dy will be required in the analysis below. Note that by definition (18), the effective wave number k of ϕ given by (22) for large α is

$$k = \alpha\left(\int_a^b g'^2|A|^2\,dy\right)^{1/2} + O(1). \qquad (23)$$

Substituting (22) into (21) and integrating by parts we obtain, for the left-hand side of (21),

$$\left.\frac{R(y,y')A(y')e^{i\alpha g(y')}}{i\alpha g'(y')}\right|_{y'=a}^{y'=b} + \frac{i}{\alpha}\int_a^b \left(\frac{A(y')}{g'(y')}\frac{\partial R(y,y')}{\partial y'} + \frac{R(y,y')A'(y')}{g'} - \frac{R(y,y')A(y')g''(y')}{g'^2}\right)e^{i\alpha g(y')}\,dy', \qquad (24)$$

where primes on g and A indicate derivatives. For flows with Dirichlet or periodic boundary conditions at $y = a$ and $y = b$, the boundary terms are zero. Otherwise, for (22) to be an eigenfunction, A must be such that the boundary terms are negligible.

We wish to use (20) to evaluate the integral in (24), but (20) is only valid for $L \gg |y' - y| \gg \eta$. Provided $\alpha\eta \ll 1$, the contribution to the integral in (24) from the region with $|y' - y|$ too small for (20) to apply is negligible. Thus, in this asymptotic analysis for $\alpha \to \infty$, we formally require that the Reynolds number increase with α fast enough so that $\alpha\eta \ll 1$ (that is, remain in the inertial range as $\alpha \to \infty$). In this case the first term in the integral is essentially singular at $y = y'$, and provided that g' is never zero, the contributions to the integral away from the singularity ($|y' - y|\alpha \gg 1$) is of higher order in $1/\alpha$. Thus, the contribution to the integral from the region with $|y' - y|$ too large for (20) to apply is also negligible. Finally, since the second two terms in the integral are not singular, they are also of higher order. We can therefore use

$$\frac{\partial R(y,y')}{\partial y'} \approx -\frac{4}{9} C\epsilon^{2/3}(\tilde{y})\delta|\delta|^{-4/3} \tag{25}$$

in (24), where $\delta = y'-y$. Furthermore, $A(y')$, $g'(y')$, $\epsilon(\tilde{y})$, and $e^{i\alpha g(y')}$ can be replaced by Taylor series about $y'=y$. Noting that δ is order $1/\alpha$, the dominant term in (24) is thus

$$-\frac{4iC\epsilon^{2/3}(y)}{9\alpha g'(y)} A(y)e^{i\alpha g(y)} \int_{a-y}^{b-y} \delta|\delta|^{-4/3} e^{i\alpha\delta g'(y)}\, d\delta, \tag{26}$$

where the expansion $e^{i\alpha g(y')} = e^{i\alpha g(y)} e^{i\alpha\delta g'(y)} \times (1 + i\alpha\delta^2 g''(y)/2 + \cdots)$ was used. Scaling $\alpha g'(y)$ out of the integral and considering the limit as $\alpha \to \infty$, (26) becomes

$$-\frac{4iC\epsilon^{2/3}(y)}{9\alpha^{5/3}[g'(y)]^{5/3}} A(y)e^{i\alpha g(y)} \int_{-\infty}^{\infty} \rho|\rho|^{-4/3} e^{i\rho}\, d\rho. \tag{27}$$

The integral in (27) evaluates to $2\pi i/\Gamma(\frac{1}{3})$, so to leading order the integral in (21) is

$$\frac{8\pi C\epsilon^{2/3}(y)}{9\Gamma(1/3)\alpha^{5/3}[g'(y)]^{5/3}} A(y)e^{i\alpha g(y)}. \tag{28}$$

The order of the next highest term in the expansion depends on the behavior of A and R near $y=a$ and $y=b$. For flows with Dirichlet or periodic boundary conditions at a and b for example, the next term is order $\alpha^{-8/3}$, and in the general case it is of order α^{-2}. For (22) to be an eigenfunction, the coefficient of $A(y)e^{i\alpha g(y)}$ must be independent of y, and this constraint, along with the normalization condition on g', allows g' to be determined;

$$g'(y) = \left(\frac{\epsilon(y)}{\tilde{\epsilon}}\right)^{2/5}, \quad \text{where } \tilde{\epsilon} = \left(\frac{1}{b-a} \int_a^b \epsilon^{4/5}\, dy\right)^{5/4}. \tag{29}$$

Then, $A(y)e^{i\alpha g(y)}$ is an eigenfunction (asymptotically), with the eigenvalue

$$\lambda = \frac{8\pi C\tilde{\epsilon}^{2/3}}{9\Gamma(\frac{1}{3})\alpha^{5/3}}. \tag{30}$$

Note that (29) implies that ϵ must be nonzero everywhere so that g' will be nonzero, as was required in the analysis. Since for large α, $k \propto \alpha$, the expected $k^{-5/3}$ inertial range for a one-dimensional spectrum is obtained. Furthermore, for the special case of homogeneous turbulence, with periodic boundary conditions at a and b; $\alpha = k$, $\tilde{\epsilon} = \epsilon$, $g' = 1$, and A is a constant. The standard one-dimensional energy spectrum can be recovered by adding the contributions from each of the three velocity components yielding $E_1(k) = 3\lambda/4\pi = C_2\epsilon^{2/3}k^{-5/3}$, with $C_2 = \frac{2}{3}C/\Gamma(\frac{1}{3})$, in agreement with the standard result.[2]

B. Three-dimensional eigenfunctions

The extension of the above analysis to the representation of the three-dimensional variation of the velocity in a finite domain $\mathscr{D}$ is straightforward; the minor subtleties are discussed below. Again, for simplicity, the variation of a single velocity component (say, the u_1 component) is considered. Generalization to the three-component result only requires more algebra.

As before, we seek an asymptotic eigenfunction ϕ in the WKB form,

$$\phi(\mathbf{x}) = A(\mathbf{x})e^{i\alpha g(\mathbf{x})}, \tag{31}$$

with $|\nabla g| > 0$ and the normalization $(1/V)\int_{\mathscr{D}} |\nabla g|^2\, d\mathbf{x} = 1$. Substituting this form into the integral equation (1) and considering R_{11} as given by (19), we see that it is necessary to integrate by parts three times to obtain a singularity that will dominate the integral. Thus if (31) is to be an eigenfunction, all the boundary integrals arising from the integrations by parts must either vanish or be of higher order than the remaining integral, which is expected to be of order $\alpha^{-11/3}$. In general, this places restrictions on the behavior of $A(y)$ at the boundaries of the domain. Assuming the boundary integrals are thus insignificant, the leading term in the asymptotic expansion of the integral is

$$-\frac{i}{\alpha^3} \int_{\mathscr{D}} \frac{\partial R_{11}(\mathbf{x},\mathbf{x}')}{\partial x_i' \partial x_j' \partial x_l'} \frac{\partial g}{\partial x_i'} \frac{\partial g}{\partial x_j'} \frac{\partial g}{\partial x_l'} \times |\nabla g(\mathbf{x}')|^{-6} A(\mathbf{x}')e^{i\alpha g(\mathbf{x}')} d\mathbf{x}'. \tag{32}$$

Again the small r approximation to R_{11} is substituted in (32), and the integral is evaluated (to leading order) by Taylor expansion of A, g, ϵ, and $e^{i\alpha g}$. The result is the three-dimensional analog of (27),

$$\frac{2iC\epsilon^{2/3}(\mathbf{x})}{\alpha^{11/3}|\nabla g(\mathbf{x})|^{11/3}} A(\mathbf{x})e^{i\alpha g(\mathbf{x})} \times \int_{\rho=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho^{-1/3} F(\theta,\phi) e^{i\cos(\theta)\rho} \times \sin(\theta)\, d\rho\, d\theta\, d\phi, \tag{33}$$

where $F(\theta,\phi)$ is the angular dependence of the third derivative of R_{11} [see (19)] in spherical coordinates, with the $\theta=0$ axis in the direction of $\nabla g(\mathbf{x})$. Evaluating the integrals yields

$$\frac{110\pi^2 C}{27\Gamma(\frac{1}{3})\alpha^{11/3}} \epsilon^{2/3}(\mathbf{x}) \left[|\nabla g(\mathbf{x})|^2 - \left(\frac{\partial g(\mathbf{x})}{\partial x_1}\right)^2 \right] \times |\nabla g|^{-17/3} A(\mathbf{x})e^{i\alpha g(\mathbf{x})}. \tag{34}$$

Note that if ∇g is in the x_1 direction, then the integral is zero, as required by incompressibility. For (31) to be an eigenfunction, the gradient of g must be related to ϵ, such that the coefficient of $A(\mathbf{x})e^{i\alpha g(\mathbf{x})}$ in (34) is independent of $\mathbf{x}$. This requires that ∇g not be in the x_1 direction anywhere in the domain. Finally, since $k \propto \alpha$, we obtain the $k^{-11/3}$ dependence expected for an inertial range in $\mathscr{E}(\mathbf{k})$, to which λ is analogous.

As a check, the above result (34) can be applied to homogeneous isotropic turbulence in a periodic domain, in which case $\alpha = k$, ∇g is a constant of magnitude 1, and A and ϵ are constants. In this case, by summing over the three velocity components and integrating on shells of constant k, we find $E(k) = k^2\lambda/(2\pi^2) = C_1\epsilon^{2/3}k^{-5/3}$, with $C_1 = \frac{55}{27}C/\Gamma(\frac{1}{3})$, in agreement with standard results.[2]

The analysis above shows that the inertial range in the KL spectrum of inhomogeneous turbulence is consistent with (indeed derivable from) Kolmogorov's original similarity hypotheses. This is precisely the way in which an inertial range in the Fourier spectrum of homogeneous isotropic turbulence is consistent (derivable from) the original hypotheses. The WKB form of the eigenfunctions (31) is clearly one sense in which Fourier analysis can be considered "local," as discussed in Sec. I A. Thus, the analysis in this section can be viewed as a formal version of the local spectrum arguments of Monin and Yaglom[2] and Batchelor.[3]

IV. EXAMPLE

The two-point correlation tensor was computed by Moin and Moser[14] for the numerically simulated plane channel flow at Re=3300 computed by Kim, Moin, and Moser.[15] This tensor was used to compute the KL eigenvalues and eigenfunctions,[14] which were part of the data used by Knight and Sirovich[4] in their study of the inertial range. Here, the eigenvalue spectra are reexamined in the light of the analysis of the previous two sections.

In the plane channel flow, the turbulence is homogeneous in two spatial directions (x_1 and x_3), and the x_2 domain extends from -1 to 1. The KL eigenfunctions $\phi^{(n)}$ are thus of the form

$$\phi_i^{(n)}(\mathbf{x}) = \hat{\phi}_i^{(n)}(x_2;k_1,k_3)e^{i(k_1x_1+k_3x_3)}, \tag{35}$$

with $\int_{-1}^{1}|\hat{\phi}|^2\,dy = 1$ (normalization). To define the equivalent wave number k of this eigenfunction, we need only determine

$$k_2^2 = \int_{-1}^{1}\left|\frac{\partial\hat{\phi}}{\partial x_2}\right|^2 dx_2. \tag{36}$$

Then $k^2 = k_1^2 + k_2^2 + k_3^2$, which is equivalent to (18).

Another set of eigenfunctions $\xi_i(\mathbf{x}) = \hat{\xi}_i(x_2;k_1,k_3)e^{i(k_1x_1+k_3x_3)}$ has also been computed based on the two-point correlation of the x_2 derivative of the velocity (the linear operator $\mathscr{L}$ discussed in the Introduction is $\partial/\partial x_2$). These functions are a basis for $\partial\hat{u}_i/\partial x_2$, so the velocity can be represented as an expansion in

$$\psi_i(\mathbf{x}) = \int_{-1}^{x_2}\xi_i(x_1,x_2',x_3)\,dx_2', \tag{37}$$

and k_2 is defined analogously to (32),

$$k_2^2 = \frac{\int_{-1}^{1}|\xi|^2\,dy}{\int_{-1}^{1}|\psi|^2\,dy}. \tag{38}$$

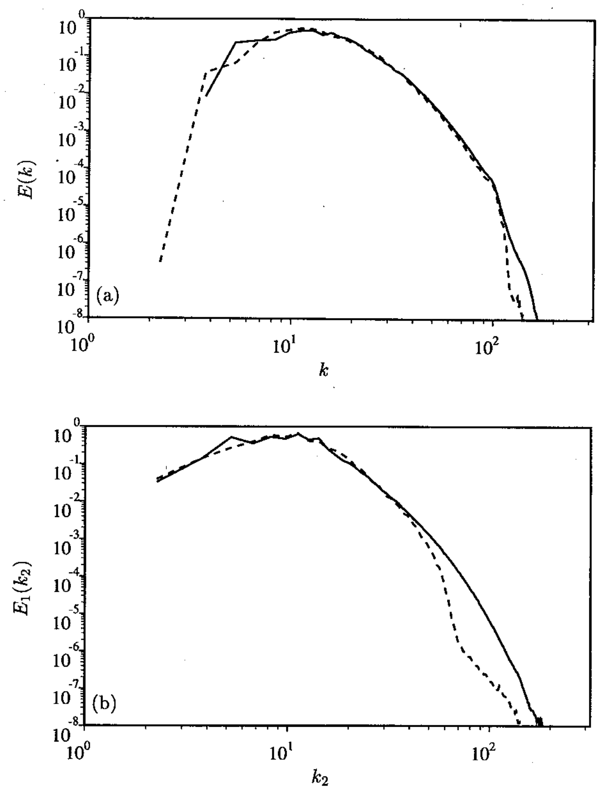

FIG. 1. (a) Three-dimensional, and (b) one-dimensional energy spectra based on (—) the energy basis; and (---) the dissipation basis.

The denominator is required here because ξ is normalized rather than ψ. An expansion in either ϕ or ψ has the properties discussed in Sec. II required to produce an inertial range (statistical orthogonality with respect to energy and dissipation). But the ψ_i basis provides an optimum representation of the dissipation, in the same way that the ϕ_i basis gives an optimum representation of the energy. In the discussion below ϕ_i and ψ_i will be referred to as the energy and dissipation bases, respectively.

Both the one-dimensional k_2 energy spectra and the three-dimensional energy spectra based on the energy and dissipation bases are shown in Fig. 1. The three-dimensional spectrum is obtained by summing the eigenvalues in shells of constant k. It is remarkable that the spectra based on the energy and dissipation bases are virtually identical over a substantial range of wave numbers. The differences are only at the largest and smallest k's. This suggests that these spectra (for intermediate k) are a measure of the flow that is in some sense independent of the details of the underlying basis (i.e., what $\mathscr{L}$ is). One wonders if the KL spectra based on $R_{ij}^{\mathscr{L}}$ would be the same for any linear invertible $\mathscr{L}$. This insensitivity of the spectra to the details of the KL basis may be due to a similarity in the bases. One might expect such a similarity if the eigenfunctions were in essence Fourier functions. The results of Sec. III suggest that this is plausible, provided k_2 is large enough to be unconstrained by the finite channel width and small enough to be unaffected by the near-wall viscous sublayer.

The other obvious feature of these spectra is that they

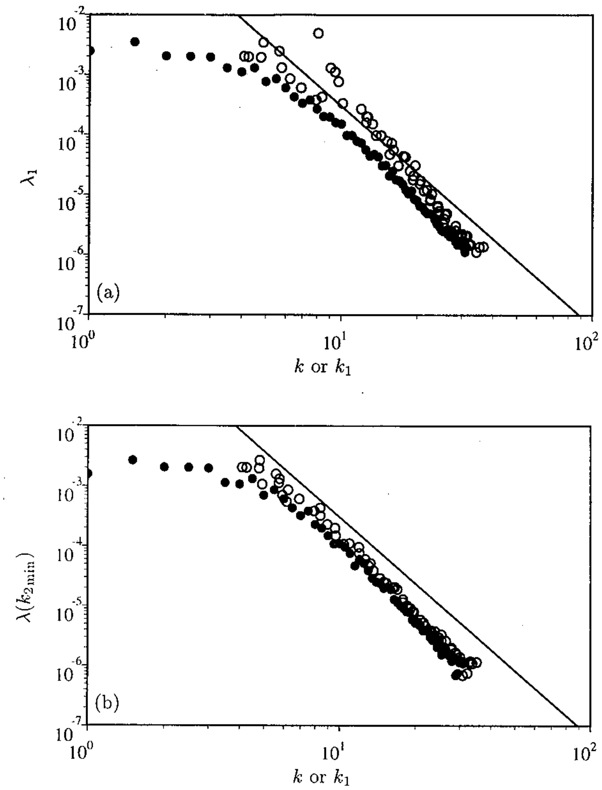

FIG. 2. Karhunen–Loeve eigenvalue spectrum (like $\mathscr{E}$) as a function of $k = \sqrt{k_1^2+k_2^2}$ (○) and k_1 (●) for the (a) maximum eigenvalue (λ_1) of each (k_1,k_3) wave number; and for the (b) eigenvalue associated with the lowest k_2 for each (k_1,k_3).

exhibit no inertial range, in agreement with the one-dimensional spectra in Ref. 15. This is to be expected because the Reynolds number is very low. The wave number at the peak in the dissipation spectrum is no more than a factor of 2 larger than the wave number at the peak in the energy spectrum.

Then how can we explain the "inertial" range observed by Knight and Sirovich?[4] One of the channel flow spectra presented by Knight and Sirovich is shown in Fig. 2(a). The values of the largest KL eigenvalue are plotted as a function of the streamwise wave number (k_1) for $k_3=0$. Also shown are the same data plotted verses the wave number k, as defined in Sec. II, which in this case is $(k_1^2+k_2^2)^{1/2}$. By comparing the two spectra, it is obvious that k_2 associated with the maximum eigenvalue is significant at the lowest wave numbers and that it varies wildly with streamwise wave number. This occurs because Knight and Sirovich selected the maximum eigenvalue associated with $k_3=0$ and each k_1, rather than selecting the eigenvalues associated with particular values of the wave number k_2. Similar spectra obtained by selecting the mode with the lowest k_2 are shown in Fig. 2(b). This produces a more consistent spectrum.

These spectra exhibit a small $k^{-11/3}$ range (a $-\frac{11}{3}$ sloped line is shown). The slope is $-\frac{11}{3}$ because these spectra represent values of $\mathscr{E}(\mathbf{k})$ along a line of approximately constant k_2 and k_3. This may also be the reason that there is an apparent inertial range when none appears in $E(k)$ or $E_1(k)$. First, suppose that a $k^{-11/3}$ range in $\mathscr{E}(\mathbf{k})$ is in some sense robust, with a short range appearing even at modest Reynolds numbers, as indicated in Fig. 2. In an inhomogeneous anisotropic flow, there is no reason to expect this short $-\frac{11}{3}$ range to appear at the same k or to the same extent for different directions of the $\mathbf{k}$ vector. Indeed, for the data analyzed here, spectra along lines in the k_2 and k_3 directions in wave space exhibit no apparent inertial range. Therefore, when $\mathscr{E}(\mathbf{k})$ is integrated in shells of constant k, the inertial range is smeared. In order to observe an inertial range in $E(k)$, the Reynolds number must be large enough so that there is an interval in k for which $\mathscr{E}$ is in the inertial range for all $\mathbf{k}$ with $|\mathbf{k}|$ in the interval. Similarly, to obtain the one-dimensional spectrum we integrate $\mathscr{E}(\mathbf{k})$ in planes of constant k_1 (say). Again a small inertial range would be smeared. Note that these arguments also suggest that in isotropic turbulence, it would be easier to see an inertial range in the three-dimensional spectrum than the one-dimensional spectra. This is easily confirmed by considering a model spectrum, as in Ref. 2.

V. CONCLUSIONS

The analysis presented here suggests that the Karhunen–Loeve (KL) eigenfunctions are a particularly good set of basis functions with which to form an inertial-range spectrum for inhomogeneous flows. Their statistical orthogonality with respect to both the energy and the dissipation allows one to define the effective wave number to be proportional to the square root of the ratio of the dissipation and energy of a mode. There is then the same relationship between the energy and dissipation spectra as with Fourier expansions. With this definition, the scaling and scale-separation arguments that lead to an inertial range Fourier spectrum in isotropic turbulence apply. In this sense, the KL eigenfunctions are the natural generalization of the Fourier functions (Fourier functions are the KL eigenfunctions in homogeneous flow directions). In addition, if we make the same local homogeneity assumptions as Kolmogorov,[1] the inertial range in the KL spectrum can be derived from Kolmogorov's[1] scaling of the structure functions. Note, however, that the local homogeneity assumptions are *not* required to make the spectral arguments for an inertial range.

In the example of the turbulent channel, it was found that, with the exception of the largest and smallest wave numbers, the KL spectra were virtually identical over a substantial range of k when based on either the velocity correlations or the velocity derivative correlations. This indicates that the KL spectrum is not very sensitive to which KL eigenfunctions are used. Such insensitivity may be due to a similarity in the form of the KL bases for sufficiently large k.

Knight and Sirovich[4] appear to have been correct in stating that the KL eigenfunctions are a particularly good basis in which to observe an inertial-range spectrum. However, this is not the reason they observed inertial-range spectra in low Reynolds number flows, when others had not. They were essentially examining the variation of the spectrum in the three-dimensional wave number space, rather than the integrated (in wave number space) spectra commonly studied (e.g., one-dimensional and three-dimensional spectra). As discussed in Sec. IV, it is likely that a small inertial range could be present in the spectra examined by Knight and Sirovich, while none would be present in the integrated spectra due to smearing by the integration.

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